Econ 237 & MgtEcon 617
Lecture 2: Incomplete Markets and Nontradable Assets (Labor Income)

Chris Tonetti Martin Schneider

Optimal choice models: a household example

- Optimal choice problem

$$\max_{\{c_t^i, b_t^i\}_{t=0}^T} E_0 \left[\sum_{t=0}^{T} \beta_i^t u_i \left(c_t^i \right) \right]$$

$$c_t^i + b_t^i = (1+r)\, b_{t-1}^i + y_t^i$$

$$b_t^i \geq b_{\min}, \quad b_{-1} \;\; \text{given}$$

- Exogenous variables (stochastic processes)
  - individual specific: household income y_t^i
  - aggregate: interest rate r
- Parameters
  - parameters describing distribution of income (y_t)
  - other individual specific parameters: β_i, $u_i\,(.)$
- Individual choices: consumption c_t^i, savings b_t^i

Role of consumption-savings problem in macro

- Building block for most quantitative models with heterog. households
  - exception: labor search—linear utility, no wealth effects
- Workhorse setup with one asset & no aggregate risk
  - time separable expected utility, income shocks, borrowing constraint
- Basic caveats
  - not well suited for asset pricing & hence dealing with wealth data
    - one common return on savings (no risk premia, all assets perfect substitutes, as in linearized REE DSGE approach
    - interest rate does depend on market structure, idiosyncratic risk
  - does not confront basic quantitative puzzles of choice under uncertainty $\rightarrow$ literature uses low risk aversion, plug-in estimation of income risk
  - not well suited to study borrowing: key source of debt in US economy is mortgages against housing collateral, does not lead to negative net worth!

Outline for Today

There are notes for today's class that fill in details. See course website.

- Income processes
  - standard approach
  - recent extensions
- Preferences
- Computation of optimal polices
  - finite-state approximations to continuous income processes
  - endogenous grid method
- Equilibrium
  - Aiyagari: production economy with asset market clearing r

Income processes

- Traditional two-step approach
  1. condition on observable characteristics from income process (age, education, gender...)
  2. model panel dynamics of "residual income"
- Step 1: for individual i of age j in period t, estimate using OLS

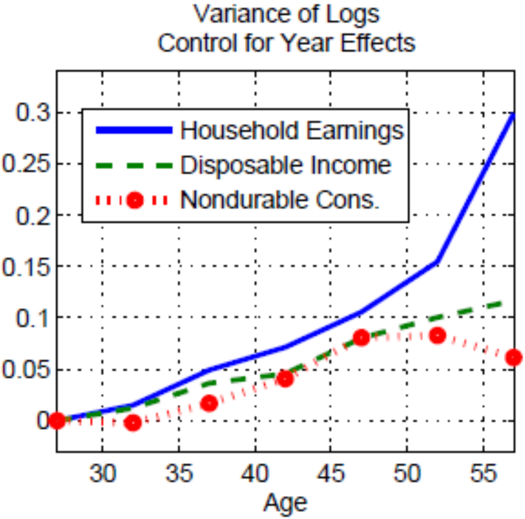
$$Y_{i,j,t} = w_t \exp(f(X_{i,j,t}) + y_{i,j,t})\bar{h}$$

$$\ln Y_{i,j,t} = \beta_t + f(X_{i,j,t}) + y_{i,j,t}$$

  - Efficiency units interpretation
  - $X_{i,j,t}$ = observable demographic characteristics
  - $y_{i,j,t}$ = idiosyncratic component
  - β_t removes aggregate component in mean
  - nonlinearity of f needed, e.g., for hump shape of income in age
- Concept of income
  - would like labor earnings, separate from capital income, sometimes difficult (entrepreneurs!)
  - hours vs. wages; with elastic labor supply can estimate idiosyncratic wage process (earnings/hours)
  - typically restrict attention to working age

Modeling residual income

- Basic facts on residual income (See Heathcote-Perri-Violante 2010)
  - cohort income fans out over life cycle → persistent component
  - but growth is not iid → transitory component (or measurement error?!)



Modeling residual income

- Step 2: standard functional form
  - define normalized age j = age–21

$$y_{i,j} = \alpha_i + \varepsilon_{i,j} + \nu_{i,j}$$
$$\varepsilon_{i,j} = \rho\varepsilon_{i,j-1} + \eta_{i,j}$$

  - fixed effect α and innovations ν, η have mean zero
  - α, ν, η all orthogonal, variances $\sigma^2_\alpha, \sigma^2_\nu, \sigma^2_\eta, \sigma^2_{\varepsilon_0}$

  $\Rightarrow$ parameters to estimate: $\rho, \sigma^2_\alpha, \sigma^2_\nu, \sigma^2_\eta, \sigma^2_{\varepsilon_0}$

- Special cases
  - $\rho = 1$: persistent component = random walk
  - $\sigma^2_\nu = 0$: capture transitory dynamics only via $\rho < 1$
  - $\sigma^2_{\varepsilon_0} = 0$: initialize only with fixed effect

Modeling residual income

- Standard Error Component Model

$$y_{i,j} = \alpha_i + \varepsilon_{i,j} + \nu_{i,j}$$

$$\varepsilon_{i,j} = \rho\varepsilon_{i,j-1} + \eta_{i,j}$$

$\Rightarrow$ parameters to estimate: $\rho, \sigma^2_\alpha, \sigma^2_\nu, \sigma^2_\eta, \sigma^2_{\varepsilon_0}$

- Time series intuition
  - autocovariance function reveals correlation structure
  - AR(1) component ε implies geometrically declining cov function
  - iid component adds extra noise at lag zero
  - same logic as inflation forecasting
- What does cross section do?
  - more power if model specification is correct
    e.g. get estimate of ρ even with short time series dimension
  - leans on common ρ for all agents!

Modeling residual income

- Standard Error Component Model

$$y_{i,j} = \alpha_i + \varepsilon_{i,j} + \nu_{i,j}$$

$$\varepsilon_{i,j} = \rho\varepsilon_{i,j-1} + \eta_{i,j}$$

$\Rightarrow$ parameters to estimate: $\rho, \sigma_\alpha^2, \sigma_\nu^2, \sigma_\eta^2, \sigma_{\varepsilon_0}^2$

$$\begin{aligned}
var\left(y_0\right) &= \sigma_\alpha^2 + \sigma_{\varepsilon_0}^2 + \sigma_\nu^2 \\
var\left(y_j\right) &= \sigma_\alpha^2 + var\left(\varepsilon_j\right) + \sigma_\nu^2; \quad j > 1 \\
var\left(\varepsilon_j\right) &= \sigma_\eta^2 \sum_{k=1}^{j} \rho^{2(j-k)} \\
cov\left(y_j, y_{j+h}\right) &= \sigma_\alpha^2 + \rho^h var\left(\varepsilon_j\right)
\end{aligned}$$

- exact identification from e.g. $var\left(y_0\right)$, $var\left(y_1\right)$, $cov\left(y_j, y_{j+1}\right)$, $cov\left(y_j, y_{j+2}\right)$, $cov\left(y_j, y_{j+3}\right)$
- leaves plenty of overidentifying restrictions!

Estimation

- Data
  - Need panel data…cross-section not enough. Why?
- Method of Moments
  - form moment conditions from above
  - weighting: optimal weighting matrix often problematic in small samples; better use equal weighting or diagonal weighting
- Standard errors
  - 2nd stage standard errors from standard MD estimator formulas
  - to incorporate 1st stage, bootstrap standard errors
    - draw say 500 bootstrap samples with replacement from data
    - run 1st & 2nd stage on each sample
    - compute stats using cross-sample variation
  - survey data with weights & multiple imputation
    - all procedures should use weights
    - surveys may supply replication weights that allow bootstrap

Income processes as model inputs: info & uncertainty

- Choice of information set
  - so far, have taken econometrician's perspective
  - for model, need to take a stand on what people know in real time
  - can they distinguish transitory & persistent components?
    - if yes, keep consumption higher after transitory shock
    - if not, filtering problem: partially mistake one shock for the other
    - over- or underreaction depending on the shock
  - do people know their fixed effect?
    - if not, response includes learning about own type...
    - overreaction to transitory shocks
    - Use data on consumption for econometrician to learn information set?
  - standard approach: persistent component & fixed effect known
    - transitory shock directly enters cash on hand
- Modeling subjective uncertainty
  - standard approach: agents are 100% sure about point estimates ("plug in estimation")
  - even if econometrician finds substantial standard error...
  - quantitative puzzles due in part to this convention ("too little risk")
  - would prefer to model estimation uncertainty; but tractability an issue

Income process as model inputs: how to deal with age?

- Traditional approach
  - use "residual income" as input for infinite horizon model
  - computationally simple; saves a state variable (age)
  - similar in structure to 1950s permanent income hypothesis
  - $\Rightarrow$ still common when focus is on income & consumption

- Consumption policy in life-cycle model
  - nonlinear in cash on hand
  - coefficients depend on age
  - can we really clean out age in stage 1 OLS regression?

Using income process as input for models: how to deal with age?

- Explicit life-cycle problems
    - same treatment of age in income process & model
    - better suited to comparisons with wealth portfolio data
    - more parameters to pick (retirement, survival probs, bequest motives?)
    - common in housing and household finance literature, increasingly used elsewhere
- A hybrid: infinite horizon with stages of life
    - e.g birth $\rightarrow$ working life $\rightarrow$ retirement $\rightarrow$ death
    - exogenous Markov chain governs stage transitions
    - calibrate chain to match average durations in each stage
    - simpler to compute: cash on hand + finite exogenous stage
    - again harder to tie to wealth data

Beyond the traditional benchmark

- HIP vs RIP
  - so far, no long term type-specific differences in profiles
  - HIP also could generate an increase in earnings variance by age
  - hard to distinguish very persistent from permanent processes (unit root time-series debates)
  - consumption behavior very different for persistent risk vs. known trend
- Higher moments
  - so far, focus on autocovariance ignores info in higher moments
  - Nonparameteric estimation: Arellano-Bonhomme (2017)
  - Complex parametric: Guvenen-Karahan-Ozkan-Song (2016)
- Distinguish wages, hours, employment, pre/post tax
  - extensive margin interacts with higher moments
- Individual income and aggregate changes
  - so far aggregate shocks only in time fixed effects
  - idiosyncratic risk is higher in bad times (cyclical)
    - See Storesletten-Telmer-Yaron and Guvenen et. al. for data variance, skewness, kurtosis
    - See Krueger-Lustig for implications for asset pricing
  - long-run trends in risk

Optimal choice models: a household example

- Optimal choice problem

$$\max_{\{c_t^i, b_t^i\}_{t=0}^T} E_0 \left[\sum_{t=0}^{T} \beta_i^t u_i \left(c_t^i \right) \right]$$

$$c_t^i + b_t^i = (1 + r)\, b_{t-1}^i + y_t^i$$

$$b_t^i \geq b_{\min}, \quad b_{-1} \ \text{ given}$$

- Exogenous variables (stochastic processes)
    - individual specific: household income y_t^i
    - aggregate: interest rate r
- Parameters
    - parameters describing distribution of income (y_t)
    - other individual specific parameters: β_i, $u_i\,(.)$
- Individual choices: consumption c_t^i, savings b_t^i

Preference parameters

- Discount factor
  - no obvious number, especially in infinite horizon approach
  - $\beta(1+r)$ regulates wealth / income ratio
- What is the curvature parameter?
  - no reason to impose IES = 1 / CRRA;
    Epstein-Zin utility already standard in finance applications
  - "$\gamma = 2$": myth propagated by (i) aggregate IES estimates &
    (ii) misguided interpretation of portfolio choice data
  - micro estimates of IES imprecise
    average is low (around .5), but higher for richer people
- More heterogeneity?
  - people differ in more than age, wealth, income
  - cognitive abilities?, family structure?, ...
  - preference heterogeneity as unobserved heterogeneity. How to estimate?
  - How to think about misspecified models/missing state variables?

Model themes

- Economic forces
  - consumption tilting according to βR
  - consumption smoothing
    - respond only to shocks, less to transitory ones
  - borrowing constraint
    - stronger response to negative shocks
- Model vs data
  - income vs wealth: too little wealth inequality
  - income vs consumption: MPCs & insurance? relative to PIH, strong response to transitory shocks, not permanent ones
    - See Blundell-Pistaferri-Preston (2008) and Kaplan-Violante (2010)
- Welfare costs of consumption fluctuations
- Secular change

Optimal choice models: a household example

- Optimal choice problem

$$\max_{\{c_t^i, b_t^i\}_{t=0}^T} E_0 \left[\sum_{t=0}^T \beta_i^t u_i \left(c_t^i \right) \right]$$

$$c_t^i + b_t^i = (1+r)\, b_{t-1}^i + y_t^i$$

$$b_t^i \geq b_{\min}, \quad b_{-1} \text{ given}$$

- Exogenous variables (stochastic processes)
  - individual specific: household income y_t^i
  - aggregate: interest rate r
- Parameters
  - parameters describing distribution of income (y_t)
  - other individual specific parameters: β_i, $u_i(.)$
- Individual choices: consumption c_t^i, savings b_t^i

Computation: Finite-State Markov Approximation of Income Process

- For simplicity, approximate $\tilde{y}$ following AR(1).
  Let $y_j \in \{y_1, y_2, \ldots, y_N\}$.

$$\tilde{y}_j = \rho\, \tilde{y}_{j-1} + \tilde{\epsilon}_j$$

- Tauchen Method
  - Pick some m, e.g., 3, s.t. $y_N = m \left(\frac{\sigma_\epsilon^2}{1-\rho^2} \right)^{0.5}$ and $y_1 = -y_N$
  - Equally space remaining y_j with distance $d := y_j - y_{j-1}$
  - Prob. shock moves y_j from y_h closest to a particular point y_k

$$\pi_{hk} := p(y_j = y_k | y_{j-1} = y_h) = p(y_k - d/2 < \rho y_h + \epsilon_j \leq y_k + d/2)$$
$$= p(y_k - d/2 - \rho y_h < \epsilon_j \leq y_k + d/2 - \rho y_h)$$

  - Let $p(\tilde{\epsilon} < \bar{\epsilon}) = G(\bar{\epsilon}) = F\left(\frac{\bar{\epsilon}}{\sigma_{\tilde{\epsilon}}}\right)$ and assume F is standard normal
  - Then for $h \in \{2, \ldots N-1\}$

$$\pi_{hk} = F\left(\frac{y_k + d/2 - \rho y_h}{\sigma_\epsilon}\right) - F\left(\frac{y_k - d/2 - \rho y_h}{\sigma_\epsilon}\right)$$

- Rouwenhorst Method (see Kopecky-Suen)
  - Better approximation for persistent processes

Computation: endogenous grid method

- Bellman equation

$$V_t(a,y) = \max_{c,a'} u(c) + \beta E_t\left[V_{t+1}\left(a',y'\right)|y\right]$$
$$c + a' = (1+r)\,a + y$$
$$a' \geq a_{\min}$$

- Suppose we have V_{t+1}, how to get V_t?
  - value function iteration: solve max problems on a grid (slow)
- Endogenous grid method: use FOC on a' grid!
  - can compute numerical derivatives for V_{t+1} on a' grid
  - solve Euler equation for each a' point: closed form with nice u!
  - find a from budget constraint: have solved problem on endog grid!
  - now need to get V_t on original grid
    - for points above lowest a, interpolate
    - for points below, get c from binding borrowing constraint
- Can also iterate on policy function using Euler equation

Equilibria in Incomplete Market Economies

Non-trivial endogenous distribution of agents across income and assets

- Aiyagari (1994) Model
  1. Income Fluctuation Problem
  2. Aggregate production function
  3. Equilibrium in asset markets (r)
- Stationary Recursive Rational Expectations Equilibria
- Transition Dynamics
- Aggregate Shocks (Krusell-Smith 1998)
- Applications: Government tax/transfer policy, optimal quantity of government debt, welfare costs of business cycles, the equity premium, etc.

Compact Asset Space: $\beta(1+r) < 1$

- Recall intertemporal and precautionary saving motive
- Intertemporal: Relation of β to $(1+r)$ important determinant of slope of consumption over time
- Precautionary saving: force favoring saving at cost of postponed consumption
- In typical income fluctuation problem, if $\beta(1+r) > 1$ patience and precautionary motive reinforce s.t. consumption and saving increase without bounds
- If $\beta(1+r) < 1$ impatience and precautionary motives compete, allowing possibility of bounded assets and consumption with ergodic distribution
- Infinitely lived vs. finite lived agents

- Demographics: Measure 1 of infinitely lived ex-ante identical agents
- Preferences: Time separable over infinite streams of consumption

$$U_t = \mathbb{E}_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau) \quad \beta \in (0,1)$$

$$u' > 0, u'' < 0$$

- Inelastic labor supply, normalized to 1 unit of time
- Endowments: Markov endowments of efficiency units z
    - $z \in Z := \{z_1, z_2, \dots z_N\}$
    - $\pi(z', z)$ transition probabilities
- Stationary distribution $\pi^*(z)$ implies constant aggregate labor supply

$$H_t = \sum_{j=1}^{N} z_j \pi^*(z_j) = H^*$$

Aiyagari 1994: Neoclassical Growth w/ Incomplete Markets

- Budget Constraint: $c_t^i + a_{t+1}^i = (1 + r_t)a_t^i + w_t z_t^i$
- Borrowing Constraint: $a_{t+1}^i \geq a_{min}$
- Technology: CRS Aggregate Production Function $Y_t = F(K_t, H_t)$, with depreciation $\delta \in (0, 1)$
- Markets: (Risk Free) (Claim on) Productive Capital K
- Resource Constraint: $C_t + K_{t+1} - (1 - \delta)K_t = F(K_t, H_t)$
- Recursive Individual Problem:

$$V(a, z) = \max_{a', c} \; u(c) + \beta \sum_{z' \in Z} \pi(z', z)V(a', z')$$

s.t.

$$c + a' = (1 + r)a + wz$$

$$a' \geq a_{min}$$

The Stationary Distribution

- Idiosyncratic State: (a, z)
- Stationary distribution of agents over $(a, z) := \lambda^*$
- State Space $S = A \times Z$; $A := [a_{min}, \bar{a}]$
- σ-algebra Σ with typical subset $\mathcal{S} = (\mathcal{A} \times \mathcal{Z})$. For any set $\mathcal{S} \subset \Sigma$, $\lambda(\mathcal{S})$ is the measure of agents in set $\mathcal{S}$
- Transition function $Q((a, z), \mathcal{S})$ is the probability an individual with current state (a, z) transits into the set $\mathcal{S}$. $Q : S \times \Sigma \to [0, 1]$

$$Q((a, z), \mathcal{A} \times \mathcal{Z}) = \mathbb{I}_{a'(a,z) \in \mathcal{A}} \sum_{z' \in \mathcal{Z}} \pi(z', z)$$

- Note: a' is optimal saving policy, so the indicator function is deterministic.

$$\lambda_{t+1}(\mathcal{S}) = \int Q((a, z), \mathcal{S}) d\lambda_t$$

- λ^*, the stationary distribution, is the fixed point of this functional equation

A Stationary Recursive Equilibrium

A Stationary Recursive Equilibrium consists of value function $v : S \to \mathbb{R}$, optimal household policies $a' : S \to \mathbb{R}$ and $c : S \to \mathbb{R}_+$, optimal firm policies H and K, wage w, rental rate r, and stationary measure λ^* such that

- Given r, w decision rule $a'(a, z)$ solves the household problem and v is the associated value function
- Given r, w, firm choices satisfy $r + \delta = F_K(K, H)$ and $w = F_H(K, H)$
- The labor market clears: $H = \int z d\lambda^*$
- The asset market clears: $K = \int a'(a, z) d\lambda^*$
- The goods market clears: $\int c(a, z) d\lambda^* + \delta K = F(K, H)$
- Stationary distribution: $\forall\ \mathcal{S} \in \Sigma, \lambda^*$ satisfies

$$\lambda^*(\mathcal{S}) = \int Q((a, z), \mathcal{S}) d\lambda^*$$

A Stationary Recursive Equilibrium

To prove existence and uniqueness, sufficient to show excess demand function (of price) in each market is continuous, strictly monotone, and crosses zero.

- Labor market is trivial: Aggregate labor supply constant H^* and labor demand decreasing in wage
- 3 Markets. By Walras law, sufficient to show equilibrium in asset market exists and is unique
- Capital Demand: $K(r) = F_k^{-1}(r + \delta)$
  - As $r \to -\delta, K \to \infty$ and as $r \to \infty, K \to 0$.
  - Demand for capital is a continuous, strictly decreasing function of r
- Capital Supply:

$$A(r) = \int a'(a, z; r) d\lambda^*(r)$$