Econ 237 & MgtEcon 617
Lectures 5 & 6: Equilibrium

Chris Tonetti   Martin Schneider

Equilibria in Incomplete Market Economies

- These notes build closely on those of Gianluca Violante and those of Victor Rios-Rull (Mpls Fed SR 231 "Computation of Equilibria in Heterogeneous Agent Models").
- These notes build on insights and results in Chamberlain and Wilson (2000), Aiyagari (1994), Kamihigashi and Stachurski (2014), Aiyagari and McGratten (1998), Floden (2001), Floden and Linde (2001), Krusell-Smith (1998), JEDC Special Issue 34(1), and Algan, Allais, Den Haan, and Rendhal Handbook of Computational Economics Chapter 6.

Equilibria in Incomplete Market Economies

Non-trivial endogenous distribution of agents across income and assets

- Aiyagari (1994) Model
    1. Income Fluctuation Problem
    2. Aggregate production function
    3. Equilibrium in asset markets (r)
- Stationary Recursive Rational Expectations Equilibria
- Transition Dynamics
- Aggregate Shocks (Krusell-Smith 1998)
- Applications: Government tax/transfer policy, optimal quantity of government debt, welfare costs of business cycles, the equity premium, etc.

Compact Asset Space: $\beta(1+r) < 1$

- Recall intertemporal and precautionary saving motive
- Intertemporal: Relation of β to $(1+r)$ important determinant of slope of consumption over time
- Precautionary saving: force favoring saving at cost of postponed consumption
- In typical income fluctuation problem, if $\beta(1+r) > 1$ patience and precautionary motive reinforce s.t. consumption and saving increase without bounds
- If $\beta(1+r) < 1$ impatience and precautionary motives compete, allowing possibility of bounded assets and consumption with ergodic distribution
- Infinitely lived vs. finite lived agents
- See Chamberlain and Wilson (2000) for rigorous analysis of compact asset space using supermartingale convergence theorem.

Aiyagari 1994: Neoclassical Growth w/ Incomplete Markets

- Demographics: Measure 1 of infinitely lived ex-ante identical agents
- Preferences: Time separable over infinite streams of consumption

$$U_t = \mathbb{E}_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau) \quad \beta \in (0,1)$$

$$u' > 0, u'' < 0$$

- Inelastic labor supply, normalized to 1 unit of time
- Endowments: Markov endowments of efficiency units z
  - $z \in Z := \{z_1, z_2, \ldots z_N\}$
  - $\pi(z', z)$ transition probabilities
- Stationary distribution $\pi^*(z)$ implies constant aggregate labor supply

$$H_t = \sum_{j=1}^{N} z_j \pi^*(z_j) = H^*$$

Aiyagari: Household Problem

- One Asset: risk-free asset with rate of return r_t
- Budget Constraint: $c_t^i + a_{t+1}^i = (1+r_t)a_t^i + w_t z_t^i$
- Borrowing Constraint: $a_{t+1}^i \geq a_{min}$
- Recursive individual problem

$$V_t(a,z) = \max_{a',c} \; u(c) + \beta \sum_{z' \in Z} \pi(z',z) V_{t+1}(a',z')$$

s.t.

$$c + a' = (1+r_t)a + w_t z$$

$$a' \geq a_{min}$$

- Histories
  - individual state variable summarizes history
  - need to know time to forecast w_t, r_t

Aiyagari: Firms & Market Clearing

- Many identical competitive firms
  - CRS production function $Y_t = F(K_t, H_t)$, with depreciation $\delta \in (0, 1)$
  - rent capital & hire labor on spot markets at prices ρ_t, w_t
  - MPs = factor prices, aggregates to representative firm because of CRS
- Absence of arbitrage: $r_t = \rho_t - \delta$
  - bonds & capital perfect substitutes
- Market clearing
  - goods, assets, labor
  - aggregate resource constraint: $C_t + K_{t+1} - (1 - \delta)K_t = F(K_t, H_t)$
- Equilibrium
  - allocations & prices s.t. households & firms optimize, markets clear
  - stationary recursive equilibrium: constant prices, stationary distribution, individual policy rules independent of time

The Stationary Distribution

- Idiosyncratic State: (a, z)
- Distribution of individual states at time t: λ_t
- State Space $S = A \times Z$; $A := [a_{min}, \bar{a}]$
- σ-algebra Σ with typical subset $\mathcal{S} = (\mathcal{A} \times \mathcal{Z})$.
- For any set $\mathcal{S} \subset \Sigma$, $\lambda(\mathcal{S})$ is the measure of agents in $\mathcal{S}$
- Transition function $Q((a, z), \mathcal{S})$ is the probability an individual with current state (a, z) transits into the set $\mathcal{S}$. $Q : S \times \Sigma \to [0, 1]$

$$Q((a, z), \mathcal{A} \times \mathcal{Z}) = \mathbb{I}_{a'(a,z) \in \mathcal{A}} \sum_{z' \in \mathcal{Z}} \pi(z', z)$$

- Note: a' is optimal saving policy, so the indicator function is deterministic.

$$\lambda_{t+1}(\mathcal{S}) = \int Q((a, z), \mathcal{S}) d\lambda_t$$

- λ^*, the stationary distribution, is the fixed point of this functional equation

Stationary Recursive Equilibrium: Definition

A Stationary Recursive Equilibrium consists of value function $v : S \to \mathbb{R}$, optimal household policies $a' : S \to \mathbb{R}$ and $c : S \to \mathbb{R}_+$, optimal firm policies H and K, wage w, rental rate r, and stationary measure λ^* such that

- Optimal choice
  - Households: Given r, w decision rule $a'(a, z)$ solves the household problem and v is the associated value function
  - Firms: Given r, w, firm choices satisfy $r + \delta = F_K(K, H)$ and $w = F_H(K, H)$
- Market clearing
  - The labor market clears: $H = \int z d\lambda^*$
  - The asset market clears: $K = \int a'(a, z) d\lambda^*$
  - The goods market clears: $\int c(a, z) d\lambda^* + \delta K = F(K, H)$
- Stationary distribution: $\forall\ \mathcal{S} \in \Sigma$, λ^* satisfies

$$\lambda^*(\mathcal{S}) = \int Q((a, z), \mathcal{S}) d\lambda^*$$

Stationary Recursive Equilibrium: Existence and Uniqueness

To prove existence and uniqueness, sufficient to show excess demand function (of price) in each market is continuous, strictly monotone, and crosses zero.

- Labor market is trivial: Aggregate labor supply constant H^* and labor demand decreasing in wage
- 3 Markets. By Walras law, sufficient to show equilibrium in asset market exists and is unique
- Capital demand (from firm FOC): $K(r) = F_k^{-1}(r + \delta)$
  - For nice F, demand for capital is a continuous, strictly decreasing function of r
  - As $r \to -\delta, K \to \infty$
  - As $r \to \infty, K \to 0$.
- Capital Supply:

$$A(r) = \int a'(a, z; r) d\lambda^*(r)$$

Existence and Uniqueness: Capital Supply

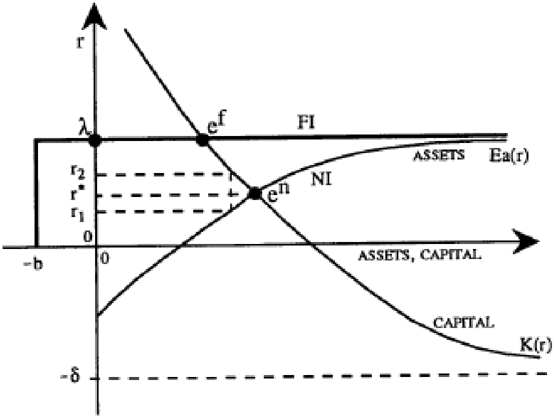
$$A(r) = \int a'(a, z; r) d\lambda^*(r)$$

- Sufficient for existence if $A(r)$ is continuous in r and crosses aggregate demand
- If $\beta(1+r) = 1$, then $A(r) \to \infty$ (Chamberlain and Wilson 2000)
- If $r = -1$, $a'(a, z) = a_{min} \quad \forall (a, z)$
- Thus, if $A(r)$ continuous, it crosses $K(r)$ and equilibrium exists
- Standard results to show $a'(a, z; r)$ is continuous and strictly increasing in r (Theorem of Maximum; see Aiyagari 1994, Huggett 1993)
- What about λ^*? First, analyze properties of equilibrium

Discussion of Equilibrium

- For intuition, think relative to complete markets benchmark
  - Recall in complete markets $1 + r = \beta$, $a_t = a_0$
  - Earnings certain or markets complete: $(1 + r)\beta < 1$, then $A(r) = a_{min}$
  - $(1 + r)\beta > 1$ implies infinite asset demand
- If $r = -1$, $a'(a, z) = a_{min} \quad \forall (a, z)$
- As $1 + r \to \beta$ from below, small increase in r implies big increase in $A(r)$
- In decision problem, by choosing $1 + r$ close to β can get arbitrarily large saving behavior
- Equilibrium r puts discipline on saving behavior, which combined with calibration can lead to quantitative insights
  - Have to discipline r to avoid arbitrary behavior
  - Other ways to do that, e.g., data on r
- Given decreasing $K(r)$ and equilibrium $1 + r < \beta$, equilibrium K must be higher than complete market K. Over accumulation of capital for self-insurance. $K^{prec} = K^{eq} - K^{FI}$
- Relaxing borrowing limit increases r by shifting $A(r)$ curve to left.

Equilibrium r: Aiyagari 1994 Figure IIb



Existence and Uniqueness: Stationary Distribution

See Kamihigashi and Stachurski (2014)
(most recent in Hopenhayn and Prescott (1992) line of research)

- SLP 12.13 says if λ^* exists and is unique, then it is continuous in r
- Existence of λ^* from compactness of S and Q with Feller property
- Uniqueness of λ^* by monotonicity of Q and Monotone Mixing Condition

1. Compactness: With $\beta(1+r) < 1$ and DARA preferences, $\exists\ \bar{a}$. Given Z, then S is compact subset of $\mathbb{R}^2$
2. Feller property of Q requires associated operator T is a mapping on functions that preserves continuity and boundedness. Easy to show since $a'(a, z)$ is continuous and bounded given compact domain of asset space.

   $T_r\lambda(\mathcal{S}) = \int Q_r((a, z), \mathcal{S}) d\lambda$

Existence and Uniqueness: Stationary Distribution

3. Monotonicity of Q: For all increasing functions $f : S \to R$, Tf is also increasing. Basically want to show conditional expectation is increasing:

$$p(a,z) := (Tf)(s) = \int f(s')Q_r(s, ds')$$

- Let π be monotone, i.e., $\mathbb{E}[f(z')|z] = \sum_{z' \in Z} f(z')\pi(z', z)$ be increasing in z. This means positive autocorrelation in the income process, an empirically valid assumption.
- Think first order stochastic dominance: $F \succ_{FSD} G \ iff \ F(x) < G(x) \ \forall \ x$. Let $\hat{a} > \tilde{a}$
  - Need to show $\sum_{z' \in \mathcal{Z}} \int_{\mathcal{A}} f(a', z')Q((\hat{a}, z), da' \times z') > \sum_{z' \in \mathcal{Z}} \int_{\mathcal{A}} f(a', z')Q((\tilde{a}, z), da' \times z')$
  - Define CDF $F(x) = \mathbb{I}_{a'(\hat{a},z) \in (-\infty, x]} \sum_{z' \in \mathcal{Z}} \pi(z', z)$
  - Define CDF $G(x) = \mathbb{I}_{a'(\tilde{a},z) \in (-\infty, x]} \sum_{z' \in \mathcal{Z}} \pi(z', z)$
  - Just compare indicator functions
  - Since a' increasing, for $\hat{a} > \tilde{a}$, $a'(\hat{a}, z) > a'(\tilde{a}, z)$ then $F \succ_{FSD} G$
- Then Q is monotone, since for a given i, j, a higher (a, z) increases the probability of being in $(a', z') > (a_i, z_j)$ next period. Thus, it is more likely to arrive in the region where f is high, since f is increasing.

Existence and Uniqueness: Stationary Distribution

4. Monotone Mixing Condition: There is a positive probability of transiting from the lowest state to some intermediate state and from the highest state to that intermediate state in finite time.
   - Suppose starting at $(\bar{a}, z_N)$, $\pi(z', z)$ stationary, and receives a long sequence of z_{min} shocks. Household receiving a long sequence of transitory negative shocks such that permanent income is above current income, try to smooth consumption via decumulating assets until hitting neighborhood of a_{min}.
   - Suppose starting at (a_{min}, z_1) and receives a long sequence of z_N shocks. Household will save, accumulating wealth since current income is higher than permanent income, until reaching some neighborhood of $\bar{a}$.

1. Conditions shown are enough to prove $A(r)$ continuous (existence). If could show $A(r)$ is strictly increasing, would have uniqueness.
2. Difficult to prove uniqueness. Depending on dominance of income and substitution effects $a'(a, z; r)$ may not be monotone in r and would still need to show effect of changes in r on λ^*.
3. No proofs of stability (convergence results from initial conditions)?

Existence and Uniqueness: Stationary Distribution

- Kamihigashi and Stachurski (2014) extend stochastic stability in monotone economies to non-compact state space. Also use weaker condition than monotone mixing condition: order reversing condition.
- Intuition: Global stability requires enough mixing. Think irreducibility of Markov chain. Otherwise can get stuck in certain regions of subspace and have multiple stationary distributions in distinct "absorbing" subsets of the state space. Also might have hysteresis.
- See also Bar Light (Economic Theory, forthcoming), who shows that uniqueness can be guaranteed if households have power utility functions with $CRRA \leq 1$ and elasticity of substitution in the production function ≥ 1. Intuition is that uniqueness comes if income effect dominates substitution effect for households. If every agent saves more when interest rates increase, then $A(r)$ is increasing in r. Can get multiplicity if $A(r)$ is ever decreasing in r (which occurs for strong enough income effects, see Acikgoz 2018 JET).

Algorithm to Compute the Equilibrium

A fixed point algorithm over r. Basically, an outer loop over the IFP that finds r such that excess capital demand is zero.

1. Guess initial iteration $r^0 \in (-\delta, \frac{1}{\beta} - 1)$
2. Given r^0, compute aggregate capital demand $K(r^0) = F_k^{-1}(r^0 + \delta)$
3. Given r^0, calculate $w(r^0)$ using CRS production function $F(K(r^0), H)$
4. Given $(r^0, w(r^0))$, solve the household IFP: $a'(a, z; r^0), c(a, z; r^0)$
5. Given $a'(a, z; r^0)$ and $\pi(z', z)$, construct transition function $Q(r^0)$
6. Given $Q(r^0)$ compute $\lambda^*(r^0)$ (next slide)
7. Compute aggregate supply of assets $A(r^0) = \int a'(a, z; r^0) d\lambda^*(r^0)$ (next slide)
8. Update to r^1
    - Define excess demand $A(r^0) - K(r^0)$ and use equation solver to find root r^*
    - If $A(r^0) > K(r^0)$, then too much desire to save, and $r^1 < r^0$. Vice versa
    - E.g., bisection method is safe and slow: for $\eta \in (0, 1)$, e.g., $\eta = 0.8$ $r^1 = \eta r^0 + (1 - \eta)\left(F_K(A(r^0), H) - \delta\right)$
    - Newton methods quicker, but require derivatives and less stable in practice

Computing $\lambda^*(r^0)$ by Simulation

Approximating infinite dimensional object. Many possible methods.

Important, small errors matter.

1. Monte Carlo simulated panel of large population of agents for given r^n
   - Good for high dimensional problems (no curse of dimensionality)
   - Bad for low dimensional problems (memory and time consuming)
2. Choose population size $I \approx 100,000$
3. Fix individual time zero (a_0^i, z_0^i) independent of r^n
4. Store sequence of shocks for each individual
   - Draw z_1^i from $\pi(z_1, z_0)$
   - Draw uniform $u \in [0,1]$. Let $z_1^i = z_{j^*}$, where

$$j^* = \arg\min_{j^*} : u \leq \sum_{j=1}^{j^*} \pi(z_j, z_0^i)$$

   - For each i update assets a'^i_{t+1} using decision rule $a'(a_t^i, z_t^i)$
5. For each t compute vector of cross-sectional moments Θ_t (mean, variance, skewness, kurtosis, percentiles). Stop when $|\Theta_{t+1} - \Theta_t|$ small
6. $A(r^n)$ is just sample mean of distribution

Computing $\lambda^*(r^0)$ by Discrete Approximation of Density

Approximate the density $\lambda^*(r^n)$ on grid of (a_k, z_j)

1. Grid to approximate density should be much finer than grid for computing optimal policy $a_k \in \{a_1, \ldots, a_M\}$
2. Start with initial density $\lambda_0(a_k, z_j)$ (e.g., uniform)
3. Given that optimal policy $a'(a_m, z_i) \in [a_k, a_{k+1}]$, randomize with probability $\frac{a_{k+1}-a'(a_m,z_i)}{a_{k+1}-a_k}$ go to a_k and complementary probability $\left(1 - \frac{a'(a_m,z_i)-a_k}{a_{k+1}-a_k}\right)$ go to a_{k+1}.
4. That is, for each (a_k, z_j):

$$\lambda_{t+1}(a_k, z_j) = \sum_{z_i \in Z} \sum_{m \in \mathcal{M}_{ik}} \pi(z_j, z_i) \frac{a_{k+1} - a'(a_m, z_i)}{a_{k+1} - a_k} \lambda_t(a_m, z_i)$$

$$\lambda_{t+1}(a_{k+1}, z_j) = \sum_{z_i \in Z} \sum_{m \in \mathcal{M}_{ik}} \pi(z_j, z_i) \frac{a'(a_m, z_i) - a_k}{a_{k+1} - a_k} \lambda_t(a_m, z_i)$$

$$\text{where} \quad \mathcal{M}_{ik} = \{m \in \{1, \ldots, M\} | a_k \leq a'(a_m, z_i) \leq a_{k+1}\}$$

5. Stop when $|\Theta_{t+1} - \Theta_t|$ small. $A(r^n) = \sum_{z_j} \sum_{a_k} a_k \lambda(a_k, z_j)$

Rough Calibration

- Production: $F(K,H) = K^{\alpha} H^{1-\alpha}$, $\alpha = \frac{1}{3}$, $\delta = 0.06$
- Preferences: $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$, $\gamma \in (1,5)$
- Insurance: Set β by targeting $\frac{K^*}{Y^*}$.
  - If $\frac{K^*}{Y^*}$ too high, too much ability to insure
  - $\frac{K^*}{Y^*} = 2?\ 4?$ Residential capital? See Koh, Santaeulalia-Llopis, Zheng (2019) about NIPA revisions and intangibles.
  - Internal calibration. Guess smaller than β^{CM}. I.e., if complete markets: $r + \delta = \alpha K^{\alpha-1} H^{1-\alpha} \rightarrow \beta^{CM} = \frac{1}{1+\alpha\frac{Y}{K}-\delta}$
  - Incomplete market β must be lower due to extra precautionary saving motive (Figure IIb)
- Borrowing limit: a_{min} set to target percent of households with zero or negative wealth (15%?)
- Too much ability to smooth consumption? See Blundell, Pistaferri, Preston (2008); Kaplan and Violante (2010)

Application: Government Taxes—Insurance vs. Efficiency

- When labor income is partially uninsurable, what is the role for gov't to provide insurance through taxes
- In Aiyagari model studied, what is optimal labor income tax?
- Adding an endogenous labor supply choice: Floden and Linde (2001)
- $u(c, \ell)$, leisure $\ell \in (0, 1)$. Optimal labor supply $h(a, z) = 1 - \ell(a, z)$
- Per capita lump-sum transfer t financed by proportional labor tax τ

$$c + a' = (1 + r)a + (1 - \tau)wzh + \phi$$

$$H = \int zh(a, z)d\lambda^*$$

$$T = \tau wH$$

- New features: $h(a, z)$ and tax with balanced budget requirement
- In equilibrium, $T = \phi$

Application: Government Taxes—Insurance vs. Efficiency

- What is the optimal level of government redistribution via taxation?
- What is the welfare gain from this government provided insurance.
- Find optimal tax rate subject to the allocations being those of a competitive equilibrium (Ramsey tax problem)
- Welfare function? One option is equal-weight social welfare function

$$\max_{\tau} \; W(\tau) = \int u(c(a,z;\tau), 1 - h(a,z;\tau))d\lambda^*$$

- Intuition: τ to small, too much variation in individual consumption. τ too big, lots of insurance, but too much disincentive to work and lower level of income. Thus $\exists \; \tau^* \in (0,1)$.

Application: Government Taxes—Insurance vs. Efficiency

- Calibrated to U.S. economy, find $\tau^* = 0.46$, and welfare gain of $\approx 9\%$ relative to $\tau = 0$
- In Sweden with smaller and less persistent income variance, $\tau^* = 0.27$
- Increase in τ increases Gini coefficient in wealth. Why?
- r is increasing in τ. Why?
- Increase in τ has bigger effect on Y in U.S. There is higher elasticity in a' and h w.r.t. τ when variance of z shocks is higher. This leads to differentially lower K, H when insurance (τ) is increased
- Capital taxes? Progressive taxes? Transfers conditioned on z?

Application: Government Bonds

Aiyagari and McGrattan (1998)

- Like Floden and Linde (2001), with optimal gov't debt B

$$T + (1 + r)B = B' + \tau w H$$

- Stationary equilibrium $B' = B$
- Government debt is risk free, so must have same r as capital
- Debt cons: Financing interest payments on debt via distortionary taxes and crowding out productive capital
- Debt pros: GE interest rate effect

$$K(r) + B = A(r) \rightarrow K(r) = A(r) - B$$

- $K(r)$ unchanged. As if aggregate asset supply shifts left by $B \rightarrow \uparrow r$
- Gov't debt enhances liquidity by providing additional means of consumption smoothing. More debt makes assets cheaper to hold, so less costly to save precautionarily

Transition Dynamics

Aiyagari + unexpected one-time permanent increase in τ

- Could compare welfare across two steady states
- Policy question: Should we raise τ? Transition to new steady state?

$$V_t(a_t, z_t) = \max_{a_{t+1}, c_t} \ u(c_t) + \beta \sum_{z_{t+1} \in Z} \pi(z_{t+1}, z_t) V_{t+1}(a_{t+1}, z_{t+1})$$

s.t.

$$c_t + a_{t+1} = (1 + r_t)a_t + w_t(1 - \tau_t)z + \phi_t$$

$$a'_{t+1} \geq a_{min}$$

Definition of Nonstationary Recursive Competitive Equilibrium

Given initial distribution λ^* and tax rates $\{\tau_t\}_{t=0}^{\infty}$ a RCE is $\{V_t\}_{t=0}^{\infty}$ and $\{c_t, a_{t+1}\}_{t=0}^{\infty}$, $\{K_t, H_t\}_{t=0}^{\infty}$, $\{r_t, w_t\}_{t=0}^{\infty}$, $\{\phi_t\}_{t=0}^{\infty}$, and $\{\lambda_t\}_{t=0}^{\infty}$ such that

1. Given $\{r_t, w_t\}_{t=0}^{\infty}$ and $\{\phi_t, \tau_t\}_{t=0}^{\infty}$, decision rules $\{c_t, a_{t+1}\}_{t=0}^{\infty}$ solve the household problem and $\{V_t\}_{t=0}^{\infty}$ are the associated value functions
2. Given $\{r_t, w_t\}_{t=0}^{\infty}$, firms choose $\{K_t, H_t\}_{t=0}^{\infty}$ optimally
3. Labor markets clear $\forall$ t: $H_t = \int z d\lambda_t = H$
4. Asset markets clear $\forall$ t: $K_t = \int a_{t+1}(a, z) d\lambda_t$
5. Goods markets clear $\forall$ t: $\int c_t(a, z) d\lambda_t + K_{t+1} - (1-\delta)K_t = F(K_t, H)$
6. Balanced Government Budget Constraint $\forall$ t: $\phi_t = \tau_t w_t H$
7. Consistent distribution: $\forall \mathcal{S} \in \Sigma, \lambda_{t+1}$ satisfies

$$\lambda_{t+1}(\mathcal{S}) = \int Q_t((a, z), \mathcal{S}) d\lambda_t$$

$$Q_t((a, z), \mathcal{S}) = \mathbb{I}_{a_{t+1}(a,z) \in \mathcal{A}} \sum_{z_{t+1} \in \mathcal{Z}} \pi(z_{t+1}, z)$$

Algorithm for Computing the Equilibrium

Exercise: $\tau_0 = \tau^*$, $\{\tau_t = \tau^{**}\}_{t=1}^{\infty}$ Assume by some finite large T economy settles to new steady state

1. Fix T (e.g., $T = 200$)
2. Compute initial steady state for $\tau = \tau^*$: $\{V^*, c^*, a^*, K^*\}$ and final steady state for $\tau = \tau^{**}$: $\{V^{**}, c^{**}, a^{**}, K^{**}\}$
3. Guess a sequence of aggregate capital $\{K_t^0\}_{t=1}^{\infty}$ s.t. $K_1^0 = K^*$ and $K_T^0 = K^{**}$
4. Given constant H:
   1. $w_t^0 = F_H(K_t^0, H)$
   2. $r_t^0 = F_K(K_t^0, H) - \delta$
   3. $\phi_t^0 = \tau_t w_t^0 H$
5. Solve optimal household policies. First, $\{V_T^0, c_T^0, a_T^0\} = \{V^{**}, c^{**}, a^{**}, K^{**}\}$. Solve by backwards induction for $\{V_t^0, c_t^0, a_{t+1}^0\}_{t=1}^{T-1}$

Algorithm for Computing the Equilibrium

6. Given optimal policies construct sequence of transition functions $\{Q_t^0\}_{t=1}^T$

7. Starting from $\lambda_0^0 = \lambda^*$, use $\{Q_t^0\}_{t=1}^T$ to generate $\{\lambda_t^0\}_{t=1}^T$

8. Given $\{\lambda_t^0, a_{t+1}^0\}_{t=1}^T$ calculate $A_{t+1}^0 = \int a_{t+1}^0(a,z)d\lambda_t^0$ (e.g., via simulation)

9. Check capital market clearing: $\max_{1\leq t\leq T} |A_t^0 - K_t^0|$ small enough
   - Note: If $|A_T^0 - K^{**}|$ small, then T was big enough

10. If capital markets don't clear, update guess of capital sequence. E.g.,

$$K_t^{n+1} = \eta K_t^n + (1-\eta)A_t^n \quad \forall\ t$$

Welfare: Conditional vs. Rawlsian

Conditional welfare comparison: Comparing utility for agent with initial state (a_0, z_0) in steady state with policy c_t^* versus in transition with policy $\tilde{c}_t$

$$V^*(a,z) = E_0\left[\sum_{t=0}^{\infty}\beta^t u(c_t^*)|(a_0 = a, z_0 = z)\right] \text{ versus}$$

$$\tilde{V}(a,z) = E_0\left[\sum_{t=0}^{\infty}\beta^t u(\tilde{c}_t)|(a_0 = a, z_0 = z)\right]$$

Consumption equivalent variation: "How much change in c in every state in stationary equilibrium to be indifferent to living in transition economy?"

$$E_0\left[\sum_{t=0}^{\infty}\beta^t u\left((1+\omega(a_0,z_0))c_t^*\right)\right] = E_0\left[\sum_{t=0}^{\infty}\beta^t u(\tilde{c}_t)\right]$$

$$\text{If } u = \frac{c^{1-\gamma}}{1-\gamma}: \ (1+\omega(a_0,z_0))^{1-\gamma}V^*(a,z) = \tilde{V}(a,z)$$

$$\omega(a_0,z_0) = \left[\frac{\tilde{V}(a,z)}{V^*(a,z)}\right]^{\frac{1}{1-\gamma}}$$

Welfare: Conditional vs. Rawlsian

- With conditional welfare change, can compute entire distribution of welfare changes. Political economy?
- Compare to "veil of ignorance" Rawlsian welfare function
- "Before knowing initial state, with probability drawn at random from λ^*, what is the welfare difference between being born in steady state vs. transition economy?"

$$\omega^U = \left[\frac{\int \tilde{V}(a,z) d\lambda^*}{\int V^*(a,z) d\lambda^*} \right]^{\frac{1}{1-\gamma}} - 1$$

- As highlighted by conditional welfare, some might loose and some might gain. Given a weighting function, this aggregates to total welfare change
- Tricky with overlapping generations and those not yet born
- ω^U can increase due to level effect (increase in average consumption), uncertainty effect (decrease in volatility of individual consumption path), and egalitarian effect (decrease in inequality across agents). ω^U mixes concern for risk with concern for interpersonal equality

Decomposing Welfare Gains

Under certain conditions, conditional welfare can be decomposed additively into level and uncertainty effects (Floden 2001). E.g., compare total welfare ω between steady states A and B. Simplify notation by dropping $\omega(a, z)$.

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t u \left((1+\omega) c^A \right) \right] = E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c^B) \right]$$

- Define average consumption C^j for $j = A, B$: $C^j = \int c^j(a, z)\ d\lambda^j$
- Define welfare change from levels as $(1+\omega^L)C^A = C^B$
- Define certainty equivalent consumption bundle $\bar{C}^j$:
  $\mathrm{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u\left(c^j\right) \right] = \sum_{t=0}^{\infty} \beta^t u\left(\bar{C}^j\right)$
- Define cost of uncertainty η^j as the fraction of average consumption an individual would give up to avoid uncertainty:
  $\sum_{t=0}^{\infty} \beta^t u\left((1-\eta^j)C^j\right) = \sum_{t=0}^{\infty} \beta^t u\left(\bar{C}^j\right)$
- Define welfare change from change in risk as $\omega^R = \frac{1-\eta^B}{1-\eta^A} - 1$
- If u homogeneous, then $1 + \omega = (1+\omega^L)(1+\omega^R) \approx 1 + \omega^L + \omega^R$

Aggregate and Idiosyncratic Risk: Krusell-Smith (1998)

Krusell-Smith Model: Technology

Krusell-Smith (1998): Aiyagari + Aggregate fluctuations

- Business cycles interacting with idiosyncratic risk
- Distribution becomes aggregate state variable; needed to forecast prices
- No analytical solution or existence/uniqueness/stability proofs, distribution is infinite dimensional $\rightarrow$ approximate equilibrium
- For simplicity let aggregate TFP $\zeta_t \in \{\zeta_\ell, \zeta_h\}$ and idiosyncratic productivity $z_t^i \in \{z_\ell, z_h\}$ for low and high
- Let $\pi(\zeta', z', \zeta, z) = Pr(\zeta_{t+1} = \zeta' \wedge z_{t+1} = z' | \zeta_t = \zeta \wedge z_t = z)$
  - A Markov process for the *joint* evolution of exogenous states
  - Typically z transition depends on ζ, but ζ transition independent of z
  - Easier to find a good job when exiting recession: $\pi(\zeta_b, z_g, \zeta_b, z_b) < \pi(\zeta_g, z_g, \zeta_b, z_b)$ and harder to keep a good job when entering recession $\pi(\zeta_b, z_g, \zeta_g, z_g) < \pi(\zeta_g, z_g, \zeta_g, z_g)$
- Now two idiosyncratic state variables (a, z) and two aggregate state variables (ζ, λ)

Krusell-Smith Model: Household Problem

$$V(a, z; \zeta, \lambda) = \max_{a', c} \; u(c) + \beta \sum_{z'} \sum_{\zeta'} \pi(\zeta', z', \zeta, z) V(a', z'; \zeta', \lambda')$$

s.t.

$$c + a' = (1 + r(\zeta, \lambda))a + w(\zeta, \lambda)z$$

$$a' \geq a_{min}$$

$$\lambda' = \Gamma(\zeta, \lambda, \zeta')$$

- V now depends on λ because households need to compute prices, prices depend on aggregate capital, and aggregate capital depends on the distribution of assets (recall $A(r)$ in Aiyagari)
- $\Gamma(\zeta, \lambda, \zeta')$ is the law of motion of the endogenous aggregate state
- Key issue: need equilibrium law of motion for λ to forecast prices. Tough!
- Γ depends on ζ b\c prices depend on ζ and decisions depend on prices
- Γ depends on ζ' b\c measure of workers with (z_ℓ, z_h) depends on ζ'

Krusell-Smith: The Distribution as a State Variable

Consider the Euler equation:

$$u_c((1+r(\zeta,\lambda))a + w(\zeta,\lambda)z - a'(a,z;\zeta,\lambda)) \geq$$
$$\beta\mathbb{E}\left[(1+r(\zeta',\lambda'))u_c((1+r(\zeta',\lambda'))a'(a,z;\zeta,\lambda) + w(\zeta',\lambda')z' - a'(a'(a,z;\zeta,\lambda),z';\zeta',\lambda'))\right]$$

- To solve for a', households need to forecast prices $r(\zeta',\lambda')$, $w(\zeta',\lambda')$, which depend on λ'
- Households need to know equilibrium law of motion Γ to forecast λ'
- Γ maps distributions to distributions $\rightarrow$ need for approximation

Krusell-Smith: Recursive Competitive Equilibrium

A Recursive Competitive Equilibrium is a value function V, household decision rules a', c, firm policies H, K, pricing functions r and w, and a law of motion Γ, such that

- Given pricing functions $r(\zeta, K)$ and $w(\zeta, K)$, decision rules a', c solve the household problem and V is the associated value function
- Given prices, firms choose K and H optimally

  $$r(\zeta, K) + \delta = \zeta F_K(K, H)$$
  $$w(\zeta, K) = \zeta F_H(K, H)$$

- Labor market clears: $H = \int z \, d\lambda$
  - Note H is no longer constant. Build π so ζ sufficient statistic for H, i.e, $H \in \{H_\ell, H_h\}$. Then, H can be perfectly forecasted using π. Prices depend on K/H, but H known function of ζ.
- Asset market clears: $K = \int a \, d\lambda$
- Goods market clears:

  $$\int c(a, z; \zeta, \lambda) \, d\lambda + \int a'(a, z; \zeta, \lambda) \, d\lambda = \zeta F(K, H) + (1 - \delta)K$$

- Aggregate law of motion Γ generated by exogenous π and endogenous a' satisfies

  $$\lambda'(\mathcal{A} \times \mathcal{Z}) = \Gamma(\zeta, \lambda, \zeta')(\mathcal{A} \times \mathcal{Z}) = \int Q_{\zeta', \zeta}((a, z), \mathcal{A} \times \mathcal{Z}) \, d\lambda$$
  $$Q_{\zeta', \zeta}((a, z), \mathcal{A} \times \mathcal{Z}) = \mathbb{I}_{a'(a, z; \zeta, \lambda) \in \mathcal{A}} \sum_{z' \in \mathcal{Z}} \pi(\zeta', z', \zeta, z)$$

Computational challenge

- Distribution λ as a state variable
  - infinite dimensional object λ
  - moves around in response to aggregate shocks according to Γ
  - fixed point problem: $\Gamma \to$ expectations $\to$ actions $\to \Gamma$
  - need parsimonious description of λ, Γ

- Approaches based on Krusell-Smith
  - describe λ by finite set of moments
  - polynomial function for Γ
  - verify fixed point by simulation

- Approaches based on Reiter: perturbation around Aiyagari
  - describe λ by histogram
  - linear law of motion of histogram in aggregate shocks
  - fixed point by method of undetermined coefficients

Krusell-Smith: Algorithm for Computation of Approximate Equilibrium

- No results on existence of a recursive competitive equilibrium with aggregate states just current TFP and distribution over idiosyncratic states.
- See Miao (JET 2006) and Cao (JET 2020) for results with RCE with extended state space, including distribution of expected discounted utilities
- Many different numerical methods to compute K-S type approximate equilibria
  - Special issue: Journal of Economic Dynamics and Control
  - Volume 34, Issue 1, Pages 1-100 (January 2010)
  - Computational Suite of Models with Heterogeneous Agents: Incomplete Markets and Aggregate Uncertainty
  - Edited by Wouter Den Haan, Ken Judd and Michel Juillard
- Terry (JMCB 2017): comparison of methods in firm problems

Krusell-Smith: Computing an Approximate Equilibrium

- Insight: Any distribution can be characterized by all of its moments (maybe infinitely many)
- Approximate infinite dimensional distribution with finite set of moments
- Let Θ be a vector of the first M moments of the *wealth* distribution, i.e., the marginal of λ w.r.t. a
- New state is $\{\theta_1, \theta_2, \ldots, \theta_M\}$ with L.o.M $\Theta' = \Gamma_\Theta(\zeta, \Theta)$
  - No longer have dependence on ζ' since only mapping wealth distribution
- Interpretation as limited information/bounded rationality
- Need to pick M and a functional form for $\Gamma_\Theta(\zeta, \Theta)$
- Insight: w, r are functions only of K, so $M = 1$ works extremely well here
- Specify linear (or log-linear) law of motion to forecast K' as function of K

$$K' = b^0_\zeta + b^1_\zeta K$$

Household problem replace λ as state variable with K and Γ with Γ_Θ. Dimension reduced from infinity to one. Compare to no-aggregate risk, which guessed K or transition dynamics which guessed deterministic sequence of K_t. Now guess law of motion for K_t. For generality, let M arbitrary.

1. $\Gamma_\Theta(\zeta, \Theta) = \Theta' = B^0_\zeta + B^1_\zeta \Theta$
2. Guess coefficient matrices B^0_ζ with dim $(M \times 1)$ and B^1_ζ with dim $(M \times M)$
3. $r(\zeta, \Theta) = \zeta F_K\left(\frac{\theta_1}{H(\zeta)}\right) - \delta$ and $w(\zeta, \Theta) = \zeta F_H\left(\frac{\theta_1}{H(\zeta)}\right)$
4. Solve the household problem for savings level a^* on (a, z, ζ, Θ) grid that satisfies EE using standard methods to compute a' function. Compared to Aiyagari, extra state variables Θ add grid dimensions

$$u_c((1 + r(\zeta, \theta_1))a + w(\zeta, \theta_1)z - a^*) \geq$$

$$\beta \sum_{\zeta', z'} (1 + r(\zeta', \Gamma_1(\zeta, \Theta)))\; u_c[(1 + r(\zeta', \Gamma_1(\zeta, \Theta)))a^* + w(\zeta', \Gamma_1(\zeta, \Theta))z' - a'(a^*, z'; \zeta', \Gamma(\zeta, \Theta))]$$

5. Simulate panel for N individuals for T periods
   - Draw sequence of aggregate shocks
   - Draw sequence of idiosyncratic shocks for each i conditional on ζ sequence
   - Use decision rules to generate panel of assets a_t^i.
   - Compute Θ for each t using sample cross-sectional moments
6. Discard first T^0 periods of data and run OLS regression for each ζ (e.g., if $\zeta_t \in \{\zeta_\ell, \zeta_h\}$ run two distinct regressions separating time periods when $\zeta = \zeta_\ell$ from those when $\zeta = \zeta_h$)

   $\Gamma_\Theta(\zeta, \Theta) = \Theta' = \beta_\zeta^0 + \beta_\zeta^1 \Theta$

1. If $\{\beta_\zeta^0, \beta_\zeta^1\} \neq \{B_\zeta^0, B_\zeta^1\}$, then update $\{B_\zeta^0, B_\zeta^1\}$ guess and iterate to convergence. Finding fixed point on forecasting rule $\{B_\zeta^0, B_\zeta^1\}$
2. Convergence means the approximate law of motion used by agents to forecast is consistent with the one generated in equilibrium by optimal behavior

Krusell-Smith: Near Aggregation

How good is this approximation to the full information rational expectations equilibrium?

- Measures of fit of forecasting regression, e.g., R^2
- Add additional moments. Does forecasting accuracy improve?
- K-S result is that $M = 1$ does remarkably well in this case:

$$\log K' = \begin{cases} 0.095 + 0.962 \log K & \text{for } \zeta = \zeta_h \\ 0.085 + 0.965 \log K & \text{for } \zeta = \zeta_\ell \end{cases}$$

$R^2 = 0.999998$

- Near Aggregation
  - If saving policies were linear, forecasting rule as function of K is exact: If $a'(a, z, \zeta, \lambda) = b_\zeta^0 + b_\zeta^1 a + b_\zeta^2 z$ then $K' = \int a'(a, z, \zeta, \lambda)\, d\lambda = b_\zeta^0 + b_\zeta^1 K + b_\zeta^2 H_\zeta = \tilde{b}_\zeta^0 + b_\zeta^1 K$
  - Away from borrowing constraint, nearly linear saving policies
  - Wealthy have more weight in determining aggregate wealth
  - Aggregate shocks do not significantly redistribute wealth across agents

Accuracy of Approximated Law of Motion

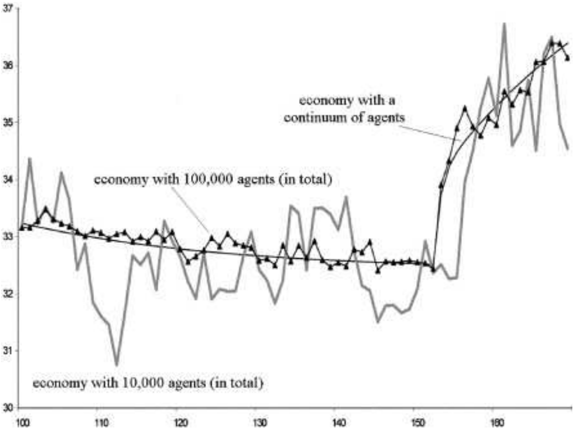
- Solutions with very high R^2 can still be inaccurate
- Want to asses accuracy of $K_{t+1} = b^0_{\zeta_t} + b^1_{\zeta_t} K_t$
- Estimated to be best linear fit to time-series of average capital K^*_t of a panel generated by optimal decision rules and exogenous ζ sequence
- Key: K^*_{t+1} and K^*_t only related through decision rules, not directly through previous law of motion
- Define $u_{t+1} = K^*_{t+1} - K_{t+1}$, with K^*_{t+1} as simulated and K_{t+1} predicted based on law of motion at time t
- Then, $u_{t+1} = K^*_{t+1} - (b^0_{\zeta_t} + b^1_{\zeta_t} K^*_t)$, since each period starts with the true simulated value and evaluates the one step ahead forecast error starting from the truth
- Error defined this way understates problems with forecasting since it doesn't allow errors to propagate across time. It always starts back on track with the true K^*

Alternatives to Standard Accuracy Tests

- A bad approximating law of motion pushes observations away from truth each period.
- Want to allow errors to accumulate for a true measure of accuracy of approximating LoM.
- Alternative: $\tilde{u}_{t+1} = K^*_{t+1} - (b^0_{\zeta_t} + b^1_{\zeta_t} K_t)$
- I.e., allow forecast errors to compound
- Also, report max error instead of mean squared error
- Can plot K^* vs. approximated sequence to see where forecast is worst
- R^2 much lower for ΔK_t
- Instead of one-step ahead forecast error, compute τ-step ahead forecast errors

Sampling Variation: Wealth per Person of Unemployed

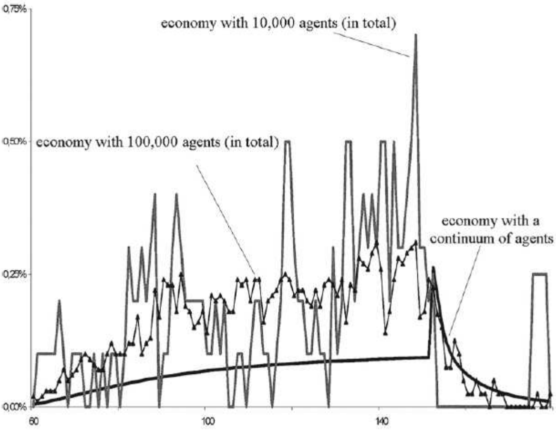
Algan, Allais-Den Haan-Rendhal, Handbook of Comp Econ chapter 6



- $\zeta = \zeta_\ell$ until $t = 155$ when $\zeta = \zeta_h$
- Not many unemployed agents, so sampling error can be large even with large simulated population

Sampling Variation: Fraction of Borrowing-Constrained People

Algan, Allais-Den Haan-Rendhal, Handbook of Comp Econ chapter 6



- $\zeta = \zeta_\ell$ until $t = 155$ when $\zeta = \zeta_h$
- With continuum, fraction of constrained agents increases as move from recession to boom

Krusell-Smith: Market Clearing, Prices, and Predetermined States

- If prices are functions of marginal product of aggregate state variable, iterating over law of motion for the aggregate state yields prices consistent with market clearing
- Given sequence of ζ_t and initial K_t, prices at t depend only on known K_t
- Individual saving policies do not affect prices at t and by construction they aggregate into K_{t+1} predicted by law of motion. Thus, asset markets at $t+1$ clear. Prices at $t+1$ are again equal to marginal product of K_{t+1}
- Market clearing is trivial because prices depend only on predetermined states
- If decision at t affected prices at t, more complicated
    - Endogenous labor supply
    - Risk-free bond
    - Housing

Reiter Method - Perturbation Approach

Approximating functions

- Functions $f(x;\theta)$
  - in application: x varies, θ fixed
  - want approximating function $\hat{f}(x;\theta)$ summarized by finite vector
- Perturbation
  - Taylor's theorem:
  - around a point x^*

$$\hat{f}(x;\theta) = f(x^*;\theta) + f_x(x^*;\theta)'(x - x^*) + ...$$

  - around a parameter value θ^*

$$\hat{f}(x;\theta) = f(x;\theta^*) + f_\theta(x;\theta^*)'(\theta - \theta^*) + ...$$

- Projection
  - global approximation
  - linear space of functions with basis $\{\phi_n(x)\}$
  - find linear combination of basis functions closest to $f(x;\theta)$

Perturbation

- Compute $\sqrt{26}$

$$\sqrt{26} = \sqrt{25(1+0.04)} = 5 * \sqrt{1.04} \approx 5 * 1.02 = 5.1$$

- Exact solution is 5.099
- Idea: solve easier problem without approximation and add small noise

Perturbation

- General idea: looking for f s.t. $T(f) = 0$
- Build family of problems indexed by scalar ε:

$$\tilde{T}(f(\varepsilon), \varepsilon) = 0$$

  - $\tilde{T}(f(1), 1) = T(f)$, so $\tilde{T}(f(1), 1) = 0$ is our original problem of interest
  - $\tilde{T}(f(0), 0) = 0$ is an easy problem to solve
- Assumptions
  - need derivatives $\partial\tilde{T}/\partial f$, $\partial\tilde{T}/\partial\varepsilon$ & $\partial f/\partial\varepsilon$
  - regularity conditions s.t. implicit function theorem applies at $\varepsilon = 0$
- Taylor expansion

$$f(\varepsilon) = f(0) + \partial f/\partial\varepsilon(0)\,\varepsilon + \ldots$$

- Find coefficient from

$$\frac{\partial\tilde{T}}{\partial f}(f(0), 0)\,\frac{\partial f}{\partial\varepsilon}(0) + \frac{\partial\tilde{T}}{\partial\varepsilon}(f(0), 0) = 0$$

Typical system for dynamic REE

- Difference equation

$$
\begin{aligned}
f\left(X_{t-1}, X_t, Y_t, Z_t\right) &= 0 \\
E_t^i\left[g^i\left(X_{t-1}, X_t, Y_t, Y_{t+1}, Z_t, Z_{t+1}\right)\right] &= 0
\end{aligned}
$$

- State variables $s_t := \left(X_{t-1}, Z_t\right)$
- Time invariant solution: functions $X'(s)$ and $Y(s)$ st $\forall s$:

$$
\begin{aligned}
f\left(X, X'(s), Y(s), Z\right) &= 0, \\
E^i\left[g^i\left(X, X'(s), Y(s), Y\left(X'(s), Z'\right), Z, Z' \,\middle|\, Z\right]\right. &= 0,
\end{aligned}
$$

  - $n + \sum_i m_i$ functional equations in $n + \sum_i m_i$ functions
  - standard solution algebra (Blanchard-Kahn, Uhlig, Sims)

Typical system for dynamic REE

- Steady state $\bar{s}$ with $X'(\bar{s}) = X'(\bar{X}, \bar{Z}) = \bar{X}$ and

$$f\left(\bar{X}, \bar{X}, Y(\bar{X}, \bar{Z}), \bar{Z}\right) = 0,$$
$$g^{i}\left(\bar{X}, \bar{X}, Y(\bar{X}, \bar{Z}), Y\left(\bar{X}, \bar{Z}\right), \bar{Z}, \bar{Z}\right) = 0$$

- Exogenous stochastic law of motion for Z
- Family of problems: replace Z by $\bar{Z}^{1-\varepsilon} Z^{\varepsilon}$
  - $\varepsilon = 1$ is problem of interest
  - $\varepsilon = 0$ is deterministic problem
- Take derivatives of f, g with respect to X, Z, ε to find coefficients in expansion of X', Y
- Treatment of uncertainty
  - first order expansion certainty equivalent
  - second order: constant risk premia
  - third order: time varying risk premia

Reiter idea

- Large system for incomplete markets model with aggregate shocks
  - family of Euler equations (including cutoff where constraint binds)
  - transition of wealth distribution
  - X includes histogram of wealth distribution
- First order approximation
  - steady state = Ayiagari solution!
  - large coefficient matrices found by computer
- Properties
  - inherits nonlinearity in behavior towards idiosyncratic risk
  - wealth distribution moves around with aggregate risk
  - but behavior towards aggregate risk is certainty equivalent
- Application to continuous time by Moll et. al.

Model Generated Wealth Distributions and Data

- Income distribution is an input into model
- Wealth distribution is endogenously determined
- Does K-S model have good match to empirical wealth distribution? No.
- Many ways to modify model. One way: heterogeneous discounting β_i
- Krusell-Smith add stochastic β with three types
- Caroll, Slacalek, Tokuoka, and White (QE 2017) add uniform deterministic β in some range
- Does it matter that we match data or how we match data?
- What are preference parameters?

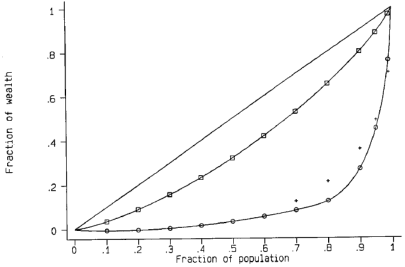


FIG. 3.—Lorenz curves for wealth holdings (+ refers to the data, □ to the benchmark model, and ○ to the stochastic-β model).